**(Full Marks: 20)**

| Part | Model Answer | Marks |
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| **A** | The physical volume is<br>$$V_p = a^3(t)V.$$ **(0.5 points).**<br>The comoving number density is a constant, thus the physical number density is<br>$$\frac{n(t)}{n(t_0)} = \left(\frac{a(t_0)}{a(t)}\right)^3.$$ **(0.5 points)**<br>The kinetic energy for non-relativistic particles are negligible, thus the energy density is<br>$$\rho_m(t) = m\,n(t),$$ **(0.5 points)**<br>where $m$ is the mass of a particle.<br>Thus<br>$$\rho_m(t) = \rho_m(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$$ **(0.5 points)**<br>**[Remarks: It is acceptable if the student just writes $\rho_m \propto 1/a^3$ and full points will be given.]** | **2** |
| **B** | The Einstein's energy relation for a massless particle is<br>$$E = pc.$$ **(0.5 points)**<br>From de Brogile's relation:<br>$$p \propto 1/\lambda_P \propto 1/a(t).$$ **(0.5 points)**<br>**[Remarks: No point if only $\lambda_p \propto a(t)$ is written because already given.]**<br>Thus<br>$$E \propto 1/a(t).$$ **(0.5 points)**<br>Physical number density is $n \propto 1/a^3$.<br>Energy density is $n\,E$.<br>Thus<br>$$\rho_r(t) = \rho_r(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4$$ **(0.5 points)**<br>**[Remarks: It is acceptable if the student just write $\rho_r \propto 1/a^4$.]** | **2** |

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| **C** | The photons in thermal equilibrium satisfy Boltzmann distribution<br><br>$$n(E(a)) \propto e^{-\frac{E(a)}{k_\mathrm{B}\,T(a)}},$$ **(1 point)**<br><br>where $E \propto 1/a(t)$.<br><br>Condition of being non-interacting implies that there is no energy transfer. Thus the energy distribution must be stable.<br><br>To be explicit, for two different comoving wavelengths,<br><br>$$\frac{n(E_1(a))}{n(E_2(a))} = e^{[E_2(a)-E_1(a)]/[k_\mathrm{B}\,T(a)]} = \mathrm{const.}$$<br><br>**[Remarks: All the above steps can be replaced by the intuition of $E \propto T$, if the students realize it, the above 1 point can be given.]**<br><br>Thus<br><br>$$T(a) \propto 1/a, \text{ i.e. } \gamma = -1.$$ **(1 point)** | **2** |
| **D** | The 1st law of thermodynamics is<br><br>$$\mathrm{d}E_\mathrm{X} = -p_\mathrm{X}\mathrm{d}V_\mathrm{p}.$$ **(1 point)**<br><br>Here no entropy term appears, because $S$ =const. No chemical potential appears, because of particle number conservation.<br><br>Here $V_\mathrm{p} = a^3 V$.<br><br>$$\mathrm{d}V_\mathrm{p} = 3a^2 V\mathrm{d}a.$$ **(1 point)**<br><br>$$E_\mathrm{X} = \rho_\mathrm{X} V_\mathrm{p}.$$ **(0.5 points)**<br><br>$$\mathrm{d}E_\mathrm{X} = \mathrm{d}(\rho_\mathrm{X} V_\mathrm{p}) = a^3 V\mathrm{d}\rho_\mathrm{X} + 3\rho_\mathrm{X} a^2 V\mathrm{d}a.$$ **(0.5 points)**<br><br>Thus<br><br>$$\mathrm{d}\rho_\mathrm{X} + 3\left(\frac{\mathrm{d}a}{a}\right)(\rho_\mathrm{X} + p_\mathrm{X}) = 0.$$ **(0.5 points)**<br><br>$$\dot{\rho}_\mathrm{X} + 3\left(\frac{\dot{a}}{a}\right)(\rho_\mathrm{X} + p_\mathrm{X}) = 0.$$ **(0.5 points)**<br><br>**[Remarks: 0.5 point for relating variation and time derivative no matter in which step it is being used.]** | **4** |

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| **E** | With lens area $A$, we only receive part of the starlight. The area ratio is<br>$$A/(4\pi a^2(t_0)r^2).$$ **(1 point)**<br>The wavelength of each photon emitted from the star gets stretched. Thus energy per photon is lowered, contributing a ratio<br>$$a(t_e)/a(t_0).$$ **(1 point)**<br>The separation among the photons also increases due to cosmic expansion, contributing a ratio<br>$$a(t_e)/a(t_0).$$ **(1 point)**<br>As a result, the power that the telescope receives is<br>$$P_r = \frac{A\, a^2(t_e)}{4\pi a^4(t_0) r^2} \times P_e.$$ **(1 point)** | **4** |
| **F** | The kinetic energy and gravitational energy of the shell adds up to a constant:<br>$$E = \frac{1}{2} m\, (\dot{r}_p)^2 - \frac{GMm}{r_p},$$ **(2 points)**<br>where<br>$$M = \frac{4\pi}{3} {r_p}^3 \frac{\rho}{c^2},$$ **(1 point)**<br>(Note: energy conservation without evolving pressure requires the assumption of non-relativistic matter.)<br>$$r_p = a(t) r,$$ **(1 point)**<br>**[Remarks: The point is given because the student understand that the shell is not pulled gravitationally from the outside, because the force due to the mass outside cancels.]**<br>Thus<br>$$\frac{2E}{mr^2} = \dot{a}^2 - \frac{8\pi G}{3c^2}\rho a^2.$$ **(1 point)**<br>***<u>Alternative Solution</u>*:**<br>For the gravitational force due to the mass inside:<br>$$m\, \ddot{r}_p = -\frac{G\,M\,m}{r_p^2} = -\frac{4\pi}{3c^2} Gm\rho r_p,$$ **(2 points)**<br>where $m$ is mass of shell. | **5** |

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| | $r_p = a(t)r$, **(1 point)**<br><br>**[Remarks: The point is given because the student understand that the shell is not pulled gravitationally from the outside, because the force due to the mass outside cancels.]**<br><br>and $\rho = \rho(t_0)a^3(t_0)/a^3(t)$.<br><br>Thus<br><br>$$\ddot{a} = -\frac{4\pi}{3c^2}G\rho(t_0)a^3(t_0)a^{-2}.$$ **(1 point)**<br><br>Integrate the above equation. One gets<br><br>$$c = \frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3c^2}\rho(t_0)a^{-1} = \frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3c^2}\rho a^2,$$ **(1 point)**<br><br>where $c$ is an integration constant. | |
| **G** | (b) decelerating. This is because gravity is attractive for the matter that we are considering here. As a result, $\mathrm{d}a(t)/\mathrm{d}t$ is a decreasing function of $t$. | **1** |
| | **Appendix: Notes about the physics behind this set of problems:**<br><br>To reduce students' reading load, we have not mentioned in the problems, that those problems set up the framework of researches in modern cosmology:<br><br>A theory of gravity (especially Einstein's general relativity) contains two aspects: Gravity tells matter how to move (kinematics of matter motion in a gravitational field); and matter determines the gravitational field (dynamics of the gravitational field). Parts (A)-(E) are about kinematics and part (F) is about dynamics in this sense. The two key equations in cosmology are derived in part (D) (this is known as the continuity equation, containing parts (A) and (B) as special cases) and (F), upon which the whole theory of modern cosmology is built.<br><br>The equation derived in part (F) is known as the Friedmann equation, which is conventionally written as $\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = \frac{8\pi G}{3}\rho$. This equation governs the dynamics of cosmic expansion and actually not only applies for non-relativistic matter but also for general matter components (which needs general relativity to derive). The constant $k$ is related to the curvature of 3-dimensional space, which is observed to be vanishingly small.<br><br>Part (C) indicates that the universe was hotter at earlier ages. The hot universe in local thermal equilibrium determines the whole thermal history of our universe, which answers questions such as where the light elements come from, and when the universe becomes transparent for light. Part (E) defines the luminosity distance, which relates the telescope observations to the cosmic reality. | **N/A** |